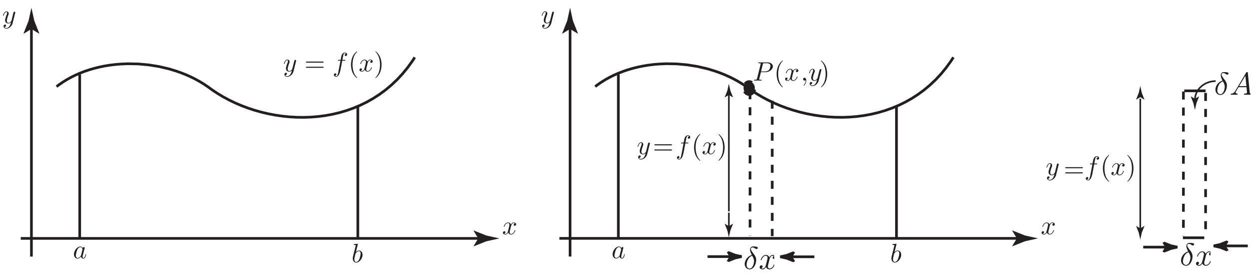
Integration as summation

Introduction

On this leaflet we explain integration as an infinite sum.

1. Integration as summation

The figure below on the left shows an area bounded by the x axis, the lines $x = a$ and $x = b$, and the curve $y = f(x)$. Note that the area lies entirely above the x axis.



There are several ways in which this area can be estimated. Suppose we split the area into thin vertical strips, like the one shown, and treat each strip as being approximately rectangular. The sum of the areas of the rectangular strips then gives an approximate value for the area under the curve. The thinner the strips, the better will be the approximation. A typical strip is shown drawn from the point $P(x, y)$. The width of the strip is labelled δx. We label it like this because the symbol δ is used to indicate a small increase in the variable being considered, in this case x. The height of the strip is equal to the y value on the curve at point P, that is $f(x)$. So the area of the strip shown is approximately $f(x)\,\delta x$. Suppose we let the area of this small strip be δA. We use the delta notation again, because this strip makes a small contribution, δA, to the total area, A, under the curve. Then

$$\delta A \approx f(x)\,\delta x$$

Now if we add up the areas of all such thin strips from a to b, which we denote by $\sum_{x=a}^{b} \delta A$, we obtain the total area under the curve.

$$\text{total area} = \sum_{x=a}^{b} \delta A \approx \sum_{x=a}^{b} f(x)\,\delta x$$

To make this approximation more accurate we must let the thickness of each strip become very small indeed, that is, we let $\delta x \to 0$, giving

$$\text{total area} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x$$

The notation $\lim_{\delta x \to 0}$ means that we consider what happens to the expression following it as δx gets smaller and smaller. This is known as the **limit of a sum**. If this limit exists we write it formally as

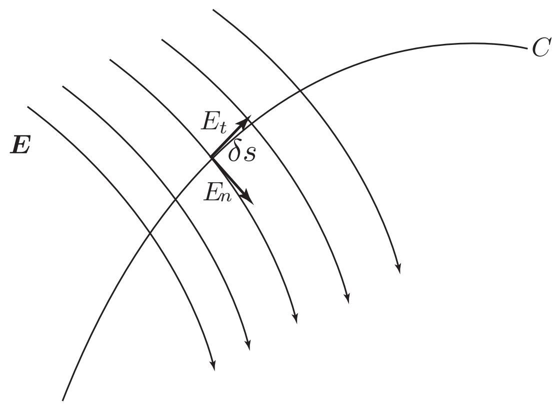
$$\int_a^b f(x)dx$$

thus defining a definite integral as the limit of a sum. Thus we have the important result that

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$$

Integration can therefore be regarded as a process of adding up, that is as a **summation**. Whenever we wish to find areas under curves, volumes etc, we can do this by finding the area or volume of a small portion, and then summing over the whole region of interest. The calculation can then be performed using the technique of definite integration.

Example



Suppose a unit charge moves along a curve C in an electric field $\mathbf{E}$. At any point on the curve the electric field vector can be resolved into two perpendicular components, E_t say, along the curve, and E_n perpendicular, or normal to the curve. In moving the charge a small distance δs along the curve the electric field does work equal to $E_t \delta s$, because only the tangential component does work. To find the total work done as the charge moves along the length of the curve we must sum all such small contributions, i.e

$$\text{total work done } = \sum E_t\,\delta s, \qquad \text{in the limit as } \delta s \to 0$$

that is

$$\text{total work done } = \lim_{\delta s \to 0} \sum E_t\,\delta s$$

which defines the integral $\int_C E_t \mathrm{d}s$. The symbol $\int_C$ tells us to sum the contributions along the curve C. This is an example of a **line integral** because we integrate along the line (curve) C.

Exercises

1. Write down, but do not calculate, the integral which is defined by the limit as $\delta x \to 0$, of the following sums.

a) $\sum_{x=3}^{x=5} 7x^2\,\delta x$, b) $\sum_{x=1}^{x=7} \frac{4}{3}\pi x^3\,\delta x$.

Answers

1. a) $\int_3^5 7x^2 \mathrm{d}x$, b) $\int_1^7 \frac{4}{3}\pi x^3 \mathrm{d}x$.

